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Momentum transport by gravity waves in a perfectly conducting shear flow

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Alfvén–gravitational waves propagating in a Boussinesq, inviscid, adiabatic, perfectly conducting fluid in the presence of a uniform aligned magnetic field in which the mean horizontal velocity $U(z)$ depends on height z only are considered. The governing wave equation has three singularities, at the Doppler-shifted frequencies $\Omega_d = 0, \pm\Omega_A$, where Ω_A is the Alfvén frequency. Hence the effect of the Lorentz force is to introduce two more critical levels, called hydromagnetic critical levels, in addition to the hydrodynamic critical level. To study the influence of magnetic field on the attenuation of waves two situations, one concerning waves far away from the critical levels (i.e. $\Omega_d \gg \Omega_A$) and the other waves at moderate distances from the critical levels (i.e. $\Omega_d > \Omega_A$), are investigated. In the former case, if the hydrodynamic Richardson number J_H exceeds one quarter the waves are attenuated by a factor $\exp\{-2\pi(J_H - \frac{1}{4})^{\frac{1}{2}}\}$ as they pass through the hydromagnetic critical levels, at which $\Omega_d = \pm\Omega_A$, and momentum is transferred to the mean flow there. Whereas in the case of waves at moderate distances from the critical levels the ratio of momentum fluxes on either side of the hydromagnetic critical levels differ by a factor $\exp\{-2\pi(J - \frac{1}{4})^{\frac{1}{2}}\}$, where J ($> \frac{1}{4}$) is the algebraic sum of hydrodynamic and hydromagnetic Richardson numbers. Thus the solutions to the hydromagnetic system approach asymptotically those of the hydrodynamic system sufficiently far on either side of the magnetic critical layers, though their behaviour in the vicinity of such levels is quite dissimilar. There is no attenuation and momentum transfer to the mean flow across the hydrodynamic critical level, at which $\Omega_d = 0$. The general theory is applied to a particular problem of flow over a sinusoidal corrugation. This is significant in considering the propagation of Alfvén–gravity waves, in the presence of a geomagnetic field, from troposphere to ionosphere.

1. Introduction

The theory of momentum transport by gravity waves in a conducting fluid in the presence of a magnetic field is an area of considerable interest in meteorological, oceanographic, geophysical and astrophysical problems. Additional interest in this field stems from the attempts to simulate solar wind–geomagnetic interactions. The examination of specific cases will serve as a check on the general theory. Recently Booker & Bretherton (1967, hereafter referred to as BB) have investigated the problem of the propagation of internal gravity waves in a shear flow with a critical level, namely a level where the mean velocity of the basic flow is equal to the horizontal phase velocity of the waves. Using linear theory and the normal-mode technique they have obtained, for an inviscid Boussinesq fluid, the wave equation

$$w_{zz} + \left[\frac{N^2}{(U-c)^2} - \frac{U_{zz}}{U-c} - k^2\right] w = 0, \tag{1.1}$$

where w is the vertical disturbance velocity, U is the basic velocity, c $(=c_r + ic_i)$ is the horizontal phase velocity, k is the horizontal wavenumber,

$$N = (-(g/\rho_0)\, d\rho_0/dz)^{\frac{1}{2}} \tag{1.2}$$

is the Brunt–Väisälä frequency, ρ_0 is the basic density and the suffix z denotes differentiation with respect to z. Equation (1.1) first appeared in the works of Taylor (1931), Goldstein (1931) and Synge (1933), and will hereafter be referred to as the TGS (Taylor–Goldstein–Synge) equation. The TGS equation (1.1) is singular at the critical level for all real phase velocities. However, by allowing the phase velocity to have a small imaginary part, and using the asymptotics of the time-dependent initial-value problem, Booker & Bretherton obtained a matching condition across the singular level which states that a wave travelling through the critical level has its amplitude attenuated by a factor $\exp[\pi(J_H - \frac{1}{4})^{\frac{1}{2}}]$, J_H being the hydrodynamic Richardson number, which is a ratio of the stabilizing effect of gravity to the destabilizing effect of shear. The linearized velocity obtained by this method eventually tends to infinity at the critical level, thus vitiating the linearization. Booker & Bretherton concluded that, once generated, the internal gravity waves may be reabsorbed by the mean flow without necessarily invoking turbulence or other dissipative processes. That is, as the wave propagates vertically through the critical level, it is strongly attenuated. The Reynolds stress, which is an appropriate measure of the magnitude of the waves, is reduced on the other side by a factor $\exp\{-2\pi(J_H - \frac{1}{4})^{\frac{1}{2}}\}$; this result was also experimentally verified by Bretherton *et al.* (1967). The analysis of Booker & Bretherton (1967) pertains to a non-rotating system. However, Jones (1967) has shown that, when the whole system rotates about a vertical axis with angular velocity Ω, for any small steady inviscid perturbation to a baroclinic shear flow the vertical flux of angular momentum is conserved and is discontinuous only across the lower and the upper critical levels, namely the levels where the Doppler-shifted frequency Ω_d $(= kU - \sigma)$ equals plus or minus the Coriolis frequency. Recently Bretherton (1969) has examined a quasi-sinusoidal wave train of inertio-gravitational waves with rotation and has given a physical picture for momentum transfer by considering the dispersion relation and vertical group velocity. He concluded that in a rotating system the process of critical-layer absorption depends only on the gross features of the situation and not on the details of the critical layer. Further, he inferred that as the wave approaches the lower or upper critical level it becomes an inertial oscillation but is not reflected, in spite of the presence of the forbidden zone where the modulus of the Doppler-shifted frequency is less than the Coriolis frequency. Detailed analysis of Jones' equation confirms this conclusion, showing that the main discontinuity in momentum flux occurs at the lowest or highest of the three singularities, depending on whether the waves are upward- or downward-propagating.

The object of the present paper is to study the internal gravity waves in a perfectly conducting shear flow with an aligned magnetic field. The basic governing differential equation of wave motion is of order two and is singular at the Doppler-shifted frequencies $\Omega_d = 0, \ \pm\Omega_A$, where $\Omega_A = Ak$ is the Alfvén frequency and A is the Alfvén velocity. That is, there are three singularities in the hydromagnetic flow whereas equation (1.1) has just one, at $\Omega_d = 0$. The layer corresponding to $\Omega_d = 0$ is the hydrodynamic critical layer and those corresponding to $\Omega_d = \pm\Omega_A$ are called magnetic critical layers. These magnetic critical layers are due to the rotational nature of the Lorentz force $\mathbf{J}\times\mathbf{B}$, which has an effect analogous to that of the Coriolis force discussed by Jones (1967).

The propagation of internal gravity waves in a perfectly conducting fluid with a transverse magnetic field has been discussed by Rudraiah & Venkatachalappa (1972*b*), and here the governing wave equation is not singular at any point in the fluid. This problem is analogous to the problem of the propagation of gravity waves in a viscous fluid studied by Hazel (1967). The corresponding problem of propagation of gravity wave in a perfectly conducting fluid in the presence of a magnetic field with a Coriolis force has been investigated recently by Rudraiah & Venkatachalappa (1972*a*). They have shown that the combined effect of the Lorentz force and the Coriolis force is to generate Alfvén–inertio-gravitational waves. The propagation of momentum from layer to layer is also discussed in detail.

The present paper yields information about the time-dependent lee waves in a perfectly conducting shear flow with an aligned magnetic field. Information is gained about the large horizontal perturbation velocities around $\Omega_d = \pm\Omega_A$. The problem of matching across the critical levels can be resolved by following the initial-value problem of BB. This matching process can also be performed by taking $c_i > 0$ and including viscosity, magnetic viscosity and heat conduction. For the hydrodynamic case, Hazel (1967) has shown that the matching condition across the critical level in the viscous problem is same as that for the inviscid initial-value problem of BB. It is of interest, for conducting flows also, to establish whether the matching condition of dissipative flows leads to the same conclusion as that for the initial-value problem of non-dissipative conducting flows and this will be presented elsewhere.

In the present paper we also try to discuss the mechanism of absorption, reflexion and transmission of waves at the critical levels. We try to discover, following Bretherton (1966), the physical explanation for absorption by examining the motion of the wave packets. It is shown that as a wave propagates vertically through the critical levels it is strongly attenuated. The algebraic sum of the Reynolds stress and the Maxwell stress, which is an appropriate measure of the magnitude of the wave, is reduced on the other side of the forbidden zone $|\Omega_d| < \Omega_A$ by a factor $\exp\{-2\pi(J-\frac{1}{4})^{\frac{1}{2}}\}$, where J is the algebraic sum of the hydrodynamic and hydromagnetic Richardson numbers. The stability of such flows has been recently investigated by Rudraiah (1970), who has shown that a sufficient condition for stability is $J > \frac{1}{4}$. In this paper we confine our attention to such stable situations.

The results of the present paper are of geophysical and astrophysical interest, one geophysical application being the study of the earth's core. Another topic of geophysical interest is concerned with the propagation of Alfvén–gravity waves from the troposphere to the ionosphere. The astrophysical application of these MHD waves has been discussed by Lighthill (1960), who dealt with both mathematical and physical aspects of the problem side by side.

2. Derivation of wave equation

To derive the wave equation for the motion of a perfectly conducting fluid in the presence of an aligned magnetic field with vertical density stratification the following approximations are made.

(i) The motion is two-dimensional, variations being in the x and z directions (i.e. horizontal and vertical directions respectively).

(ii) The fluid is inviscid, perfectly conducting and adiabatic. It is of interest to inquire as to the effects of relaxing this assumption: work relating to this is in progress. However, we note that, since the problem considered in this paper pertains to atmospheric, astrophysical and oceanic phenomena where, although the flow velocities are rather small, the Reynolds number and the magnetic Reynolds number are quite large, an inviscid perfectly conducting flow model would appear to be a reasonable one.

(iii) The Boussinesq approximation.

(iv) The rotation of the earth may be neglected. The effect of rotation on internal gravity waves in the presence of a magnetic field has been recently investigated by Rudraiah & Venkatachalappa (1972*a*).

(v) The perturbation velocity (u, w) from the basic state $U(z)$ in the x direction and the perturbation magnetic field (h_x, h_z) from the aligned uniform basic magnetic field H_0 = constant are so small that

$$\left|u\frac{\partial}{\partial x}+w\frac{\partial}{\partial z}\right| \ll \left|\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right|,$$

$$\left|h_x\frac{\partial}{\partial x}+h_z\frac{\partial}{\partial z}\right| \ll \left|\frac{\partial}{\partial t}+H_0\frac{\partial}{\partial x}\right|.$$

(vi) The total Richardson number

$$J = J_H + J_M = \left[\frac{N^2+\Omega_A^2}{(dU/dz)^2}\right] > \tfrac{1}{4} \quad \text{everywhere},$$

where $\Omega_A = kA$ is the Alfvén frequency, $A = (\mu H_0^2/\rho_0)^{\frac{1}{2}}$ being the Alfvén velocity, μ is the magnetic permeability, k is the horizontal wavenumber, $J_H = N^2/(dU/dz)^2$ is the hydrodynamic Richardson number and $J_M = \Omega_A^2/(dU/dz)^2$ is the hydromagnetic Richardson number, which is the ratio of the stabilizing effect of magnetic field to the destabilizing effect of shear.

Under these assumptions the linearized equations of motion are

$$\begin{bmatrix} \rho_0 D & \rho_0(D_2 U) & D_1 & 0 & -\mu H_0 D_1 & 0 \\ 0 & \rho_0 D & D_2 & g & 0 & -\mu H_0 D_1 \\ D_1 & D_2 & 0 & 0 & 0 & 0 \\ 0 & D_2\rho_0 & 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & D_1 & D_2 \\ -H_0 D_1 & 0 & 0 & 0 & D & (-D_2 U) \\ 0 & -H_0 D_1 & 0 & 0 & 0 & D \end{bmatrix} \begin{bmatrix} u \\ w \\ P \\ \rho \\ h_x \\ h_z \end{bmatrix} = 0, \qquad (2.1)$$

where P is the perturbed total pressure, ρ is the perturbed density, g is the acceleration due to gravity, $D = \partial/\partial t + UD_1$, $D_1 = \partial/\partial x$ and $D_2 = \partial/\partial z$. By eliminating u, P, ρ, h_x and h_z from (2.1), we obtain a single wave equation

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^4 (w_{xx} + w_{zz}) - \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^3 (U_{zz} w_x) \\ &\quad + \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \{N^2 w_{xx} - A^2(w_{xxxx} + w_{xxzz})\} \\ &\quad + A^2\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)(2U_z w_{xxxz} + U_{zz} w_{xxx}) - 2A^2 U_z^2 w_{xxxx} = 0. \qquad (2.2) \end{aligned}$$

This equation forms the starting point of the analysis of this paper. The hydromagnetic wave equation (2.2) is of order six whereas the hydrodynamic wave equation of BB is of order four.

We assume that the two-dimensional transient disturbance produced by temporary extraneous forces may be represented in the form

$$w = \mathrm{Re}\left[\frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^{+\infty} dc\, \hat{w}(k,z,c)\, e^{ik(x-ct)}\right], \qquad (2.3)$$

where each Fourier component has a well-defined horizontal wavenumber k (> 0) and phase velocity c and has a vertical structure of the form

$$\frac{d^2\hat{w}}{dz^2} + \frac{2k\Omega_A^2 U_z}{\Omega_d(\Omega_d^2 - \Omega_A^2)}\frac{d\hat{w}}{dz} + \left[\frac{k^2 N^2}{\Omega_d^2 - \Omega_A^2} - \frac{kU_{zz}}{\Omega_d} - k^2 - \frac{2k^2\Omega_A^2 U_z^2}{\Omega_d^2(\Omega_d^2 - \Omega_A^2)}\right]\hat{w} = 0. \qquad (2.4)$$

On comparing (1.1) and (2.4) we see that although both represent second-order differential equations they are quite different in the sense that the hydromagnetic wave equation (2.4) represents the telegraphic equation, having three singularities at $\Omega_d = 0, \pm\Omega_A$, whereas the hydrodynamic wave equation (1.1) represents the simple harmonic equation, having only one singularity, at $\Omega_d = 0$. Therefore the effect of the Lorentz force on the flow is to increase the number of critical levels. It is of interest to note that the appearance of $d\hat{w}/dz$ in (2.4) does not imply a damping of the wave, but rather a change in its structure such that $\hat{w}$ varies with z even though the wave strength is constant (see §5 below). For a uniform stream, with U independent of z, (2.4) reduces to

$$\frac{d^2\hat{w}}{dz^2} + \frac{k^2}{\Omega_d^2 - \Omega_A^2}(N^2 + \Omega_A^2 - \Omega_d^2)\hat{w} = 0. \qquad (2.5)$$

3. Solutions of wave equations

In this section we try to discuss the solutions of the wave equations (2.4) and (2.5) near the singular levels, namely the levels at which $\Omega_d = 0$, $\pm\Omega_A$, at moderate distances ($\Omega_d > \Omega_A$) and large distances ($\Omega_d \gg \Omega_A$) from the singular levels using the method of Frobenius. Near the critical level $\Omega_d = 0$, i.e. $U = c$, the complete solution of (2.4) is of the form

$$\hat{w} = A_1(z-z_0)^2[1+a_1(z-z_0)^2+\ldots]+B_1(z-z_0)[1+b_1(z-z_0)^2+\ldots], \quad (3.1)$$

where z_0 is such that $U(z_0) = c$, A_1 and B_1 are constants of integration and a_1 and b_1 are known constants. The complete solution of (2.4) near the upper magnetic critical level $\Omega_d = \Omega_A$, i.e. $U = c+A$, is of the form

$$\hat{w} = [A_2+B_2\log(z-z_1)][1+c_1(z-z_1)+\ldots]+B_2\sum_{k=0}^{\infty}\left(\frac{\partial c_k}{\partial r}\right)_{r=0}(z-z_1)^k, \quad (3.2)$$

where z_1 is such that $U(z_1) = c+A$ and A_2 and B_2 are constants of integration. A similar solution can be obtained near the lower magnetic critical level $\Omega_d = -\Omega_A$, i.e. $U = c-A$.

We note that solution (3.1) near the critical level $U = c$ is entirely different from the solution of the hydrodynamic equation (1.1). Equation (3.1) has no branch point at the critical level $z = z_0$ whereas the solution of (1.1) does have a branch point at $z = z_0$. However, to find the solutions away from the critical levels we assume that $\Omega_d > \Omega_A$ and $N^2+\Omega_A^2 \gg \Omega_d^2$. In this case a power-series solution for (2.4) in descending powers of $z-z_0$ may be obtained; $\zeta = 1/(z-z_0)$ is substituted into the solution and the resulting equation can be solved in a power-series expansion about $\zeta = 0$ by the method of Frobenius. The expansion is valid in the range

$$\left|\frac{k}{\Omega_A}\frac{dU}{dz}(z-z_0)\right| > 1.$$

The resulting solution, in terms of $z-z_0$, is

$$\hat{w} = A_3(z-z_0)^{\frac{1}{2}+i\mu_m}[1+d_1(z-z_0)^{-1}+d_2(z-z_0)^{-2}+\ldots]+B_3(z-z_0)^{\frac{1}{2}-i\mu_m} \times[1+e_1(z-z_0)^{-1}+e_2(z-z_0)^{-2}+\ldots], \quad (3.3)$$

where

$$\mu_m = (J-\tfrac{1}{4})^{\frac{1}{2}}, \quad d_1 = e_1 = 0, \quad d_2 = [f(\lambda)]_{\lambda=-\frac{1}{2}-i\mu_m}, \quad e_2 = [f(\lambda)]_{\lambda=-\frac{1}{2}+i\mu_m},$$

$$f(\lambda) = \frac{J_M}{k^2}\left[\frac{\lambda(3\lambda+5)}{2(2\lambda+3)}+\frac{J+1}{2\lambda+3}\right].$$

To find the solution far away from the critical levels we assume that

$$N \gg |\Omega_d| \gg \Omega_A.$$

The solution is then given by

$$\hat{w} = A_4(z-z_0)^{\frac{1}{2}+i\mu_0}+B_4(z-z_0)^{\frac{1}{2}-i\mu_0}, \quad (3.4)$$

where $\mu_0 = (J_H-\frac{1}{4})^{\frac{1}{2}}$, J_H ($>\frac{1}{4}$) being the hydrodynamic Richardson number. Thus the solution is independent of magnetic field. Therefore, far away from the critical levels the effect of the magnetic field on waves is negligible. Hence it is clear that the solutions to the hydromagnetic system approach asymptotically those of the hydrodynamic system given by BB only when the relation

$$N \gg |\Omega_d| \gg \Omega_A$$

is satisfied. In zones where this condition does not hold, the hydromagnetic and hydrodynamic systems yield widely differing solutions which, however, converge on either side of the zone. Similar behaviour is observed in the case of propagation of internal gravity waves in a rotating system discussed by Jones (1967). Therefore the behaviour of solutions in both non-rotating and rotating hydrodynamic systems far away from the critical layers is almost equivalent.

We note that the omission of higher order terms in the power series in (3.4) does not modify the structure of wave far away from the critical levels. If c is complex, i.e. $c = c_r + ic_i$, then the critical level z_c is defined by

$$U(z_c) - c_r = 0.$$

In this case the solution near the critical level is given by (3.1) with z_0 replaced by $z_c + ic_i/U_z$. Solutions (3.3) and (3.4), which are valid at moderate and at large distances from the critical levels respectively, are similar to that of the hydrodynamic equation (1.1) near the critical level $\Omega_d = 0$. These solutions are also similar to one obtained by Rudraiah & Venkatachalappa (1972*a*) for a rotating system.

Now, if we fix the branch of the complex powers in (3.3) by taking

$$(z-z_0)^{\frac{1}{2}\pm i\mu_m} = |z-z_0|^{\frac{1}{2}}\, e^{\pm i\mu_m \log|z-z_0|} \quad \text{if} \quad z > z_0, \tag{3.5}$$

it then follows that

$$(z-z_0)^{\frac{1}{2}\pm i\mu_m} = -i\, e^{\pm\mu_m\pi}\, |z-z_0|^{\frac{1}{2}}\, e^{\mp i\mu_m \log|z-z_0|} \quad \text{if} \quad z < z_0. \tag{3.5a}$$

The magnitude of each term in (3.3) at a given distance above the critical level $\Omega_d = 0$ is not the same as that at the same distance below but differs by a factor $\exp(\pm\mu_m\pi)$ however small c_i may be, provided that $c_i > 0$. This clearly shows that the effect of the Lorentz force at the critical levels is to increase the attenuation or to decrease the amplitude by a factor of J_M relative to the hydrodynamic results of BB. However, for waves far away from the critical levels the attenuation factor, from (3.4), is $\exp\{-\mu_0\pi\}$, which is exactly the same as the corresponding hydrodynamic result of BB. To discover the nature of the wave near the critical evels we use (3.1) and (3.2). The magnitude of each term in (3.1) is the same at a given distance above or below $\Omega_d = 0$. However, although the magnitude of the first term in (3.2) is the same at a given distance above or below the critical level $\Omega_d = \Omega_A$, the second term is not, the square of the magnitude differing by an amount $|B_2|^2\pi^2$. Thus the square of the amplitude of the waves is decreased on either side by an amount $|B^2|^2\pi^2$.

The solution of (2.5), which is the case of uniform basic flow, is

$$\hat{w} = C\, e^{imz} + D\, e^{-imz}, \tag{3.6}$$

where C and D are constants of integration and

$$m = k[N^2/(\Omega_d^2 - \Omega_A^2) - 1]^{\frac{1}{2}}. \tag{3.7}$$

For the sake of definiteness we settle the branch for m by requiring that

$$m_i > 0 \quad \text{if} \quad c_i > 0. \tag{3.8}$$

To study the nature of m we have considered, separately, the two situations $\Omega_d^2 < \Omega_A^2$ and $\Omega_d^2 > \Omega_A^2$. If $\Omega_d^2 < \Omega_A^2$, i.e. the Doppler-shifted frequency is less than the Alfvén frequency, m is imaginary irrespective of the magnitude of N and $m = ikm_1$, where

$$m_1 = \left[\frac{N^2}{\Omega_A^2 - \Omega_d^2} + 1\right]^{\frac{1}{2}}. \tag{3.9}$$

In this case the complete spatial distribution of velocity associated with the first solution in (3.6) is

$$w = \mathrm{Re}\left[C\, e^{-km_1 z} \exp\{ik(x - ct)\}\right], \tag{3.10}$$

which represents a plane wave of variable amplitude, diminishing with height z, and with a phase front (for real c) given by $kx - kct =$ constant. The second solution $\hat{w} = D\, e^{km_1 z}$ can be interpreted similarly.

The situation $\Omega_d^2 > \Omega_A^2$, i.e. when the Doppler-shifted frequency is greater than the Alfvén frequency, is quite different. This implies that if $N^2/(\Omega_d^2 - \Omega_A^2) \gg 1$

$$m \sim \begin{cases} +\dfrac{kN}{(\Omega_d^2 - \Omega_A^2)^{\frac{1}{2}}} & \text{if} \quad \Omega_d > 0, \\ -\dfrac{kN}{(\Omega_d^2 - \Omega_A^2)^{\frac{1}{2}}} & \text{if} \quad \Omega_d < 0, \end{cases} \tag{3.11}$$

and if $N^2/(\Omega_d^2 - \Omega_A^2) \ll 1$

$$m \sim ik. \tag{3.12}$$

Signs in (3.11) and (3.12) are chosen to satisfy (3.8).

4. Upward- and downward-propagating waves

The phenomenon of absorption, transmission and reflexion of momentum at the critical levels will depend on whether the waves are upward- or downward-propagating. Therefore in this section we try to interpret solutions discussed in § 3 as upward- or downward-propagating waves. As in the hydrodynamic case (BB), in magnetohydrodynamics it is difficult, in a non-uniform medium, to specify which part of an oscillatory motion corresponds to a wave travelling in the upward direction and which to a wave travelling downwards since there is a continual interchange between the two. In a uniform media, on the other hand, precise and physically important identifications may be made. Therefore the cases of uniform and non-uniform media are discussed separately. In this section we consider only the uniform media; the case of non-uniform media will be discussed in the next section.

For the case of uniform media, the solution is given by (3.6). From (3.11), we find that if Ω_d is negative m is also negative, so that the phase fronts move downwards, if Ω_d is positive m is also positive. Thus the first solution in (3.6) describes a wave with a downward component of phase velocity. However, the influence of such a wave propagates upwards so it is called an upward-travelling wave.

The second solution will similarly be interpreted as a downward-travelling wave. The significance of these waves, as in hydrodynamics (BB), can be interpreted in the following three ways.

(i) *Group velocity approach.* This dispersion relation, from (3.7), is

$$\sigma = kU \pm \{k^2A^2 + k^2N^2/(m^2+k^2)\}^{\frac{1}{2}}. \tag{4.1}$$

According to (3.11) we must take the minus sign when m and Ω_d are positive and the plus sign when they are negative. In either case, for the first solution in (3.6)

$$\partial\sigma/\partial m = m(\Omega_d^2 - \Omega_A^2)^2/k^2N^2\Omega_d, \tag{4.2}$$

which is always positive and hence corresponds to an upward component of group velocity. Thus the first solution in (3.6) represents an upward-propagating wave and similarly the second solution represents a downward-propagating wave.

(ii) *Energy approach.* A second way of understanding upward- and downward-moving waves comes from energy consideration. The total mean rate of work done by the conducting fluid, in the presence of the magnetic field, below any level on the fluid above is $\overline{Pw}$, where P is the total disturbance pressure due to hydrodynamic and hydromagnetic pressures and an overbar denotes the time average. The horizontal momentum equation for the disturbance, namely

$$(U-c)\frac{\partial u}{\partial x} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + \frac{\mu H_0}{\rho_0}\frac{\partial h_x}{\partial x}, \tag{4.3}$$

shows that

$$\overline{Pw} = -(U-c)\,[\rho_0\overline{uw} - \mu\overline{h_x h_z}]. \tag{4.4}$$

This, for the first solution in (3.6), takes the form

$$\overline{Pw} = (\rho_0 m|c|^2/k^2\Omega_d)\,(\Omega_d^2 - \Omega_A^2),$$

which is always positive since $\Omega_d^2 > \Omega_A^2$. Thus the wave energy is flowing upwards. In the case of second solution, on the other hand, the flow of energy is downwards. We note that there will be an upward total horizontal momentum flux represented by $\rho_0\overline{uw} - \mu\overline{h_x h_z}$ coupled with this flow of energy, where by the total momentum flux we mean the algebraic sum of the momentum fluxes in the fields and the material media. From this it is clear that the momentum is transferred not only by the material media but also by the magnetic field.

(iii) *Slightly complex horizontal phase velocity.* The third way of seeing that the first solution of (3.6) represents an upward-travelling wave is by considering c to be slightly complex ($c_i > 0$). Because of (3.8) the solution $C\,e^{imz}$ tends exponentially to zero as $z \to \infty$. Thus the wave amplitude at every point increases with time, but at any fixed time decreases as z is increased. Thus the variations in amplitude move upwards. Hence the first solution corresponds to an upward-travelling wave. Similarly the second solution represents a downward-travelling wave.

5. Absorption of waves near the critical levels

In the case of uniform media discussed in §4 the wave equation (2.5) is singular throughout the region of interest if $\Omega_d = \Omega_A$ as Ω_d is constant. Hence we cannot deal with the transfer of momentum to the mean flow. However, in the case of non-uniform media the wave equation (2.4) is singular at $\Omega_d = 0, \pm\Omega_A$, which defines distinct critical levels. In this case we can discuss the transfer of momentum to the mean flow. The absorption of waves near the critical levels is discussed using (*a*) momentum transfer to the mean flow and (*b*) the group velocity approach.

5.1. *Transfer of momentum to the mean flow*

In this section we discuss the transfer of momentum near to, at moderate distances from and far from the critical levels and the interpretation of upward- and downward-travelling waves. We find that the vertical flux of mean horizontal momentum is given by

$$\overline{\rho_0 uw} - \mu\overline{h_x h_z} = \mathrm{Re}\left\{\frac{i\rho_0}{k\Omega_d^2}(\Omega_d^2 - \Omega_A^2)\, w^* \frac{dw}{dz}\right\}, \tag{5.1}$$

where w^* is the complex conjugate of w. By differentiating (5.1) with respect to z and using the wave equation (2.4) we get

$$\frac{d}{dz}[\rho_0\overline{uw} - \mu\overline{h_x h_z}] = 0. \tag{5.2}$$

Hence the total upward momentum flux is conserved everywhere except at the critical levels, where the substitution of the wave equation (2.4) is invalid. Thus the upward transfer of momentum by the algebraic sum of the Reynolds stress and the magnetic stress has zero divergence, and there can be no transfer to the mean flow. Hence, either the momentum flux or the total stress $(\overline{uw} - (\mu/\rho_0)\overline{h_x h_z})$ can be taken as the measure of the strength or magnitude of the wave.

At the critical level $\Omega_d = 0$, equation (5.1), using (3.1), becomes

$$\rho_0\overline{uw} - \mu\overline{h_x h_z} = \mathrm{Re}\left\{\frac{i\rho_0}{k^3 U_z^2}[\Omega_A^2(2A_1 B_1^* + A_1^* B_1^*) + \ldots]\right\}, \tag{5.3}$$

which has a constant leading term. In other words, the momentum is continuous across the critical level $\Omega_d = 0$. This is contrary to the hydrodynamic result of BB, where it was shown that the momentum flux is discontinuous across the critical level $\Omega_d = 0$. Near the critical level $\Omega_d = \Omega_A$ the total momentum flux is given by

$$\rho_0\overline{uw} - \mu\overline{h_x h_z} = \left\{\begin{array}{ll} \mathrm{Re}\left\{\dfrac{2iA_2^* B_2}{\Omega_A}\rho_0\dfrac{dU}{dz}\right\} & \text{when} \quad z > z_1, \\ \mathrm{Re}\left\{\dfrac{-2\rho_0 dU/dz}{\Omega_A}[iBA_2^* + |B_2|^2\pi]\right\} & \text{when} \quad z < z_1. \end{array}\right\} \tag{5.4}$$

This is essentially discontinuous across the critical level $\Omega_d = \Omega_A$. Similarly we can show that the total horizontal momentum flux is discontinuous across the critical level $\Omega_d = -\Omega_A$.

The total momentum flux at moderate distances from the critical levels is

$$\rho_0\overline{uw}-\mu\overline{h_x h_z}=\begin{cases}-\dfrac{\rho_0\mu_m(\Omega_d^2-\Omega_A^2)}{2k\Omega_d^2}[|A_3|^2-|B_3|^2] & \text{when} \quad z>z_0,\\[2ex] +\dfrac{\rho_0\mu_m(\Omega_d^2-\Omega_A^2)}{2k\Omega_d^2}[|A_3|^2e^{2\mu_m\pi}-|B_3|^2e^{-2\mu_m\pi}] & \text{when} \quad z<z_0.\end{cases} \tag{5.5}$$

From this it is clear that the magnitudes of each term in (5.5) at a given distance above and below the critical level $\Omega_d = 0$ are not the same but differ by a factor $\exp\{-2\mu_m\pi\}$. For the first solution $A_3(z-z_0)^{\frac{1}{2}+i\mu_m}$ the energy flux

$$\overline{Pw} = -(\Omega_d/k)[\rho_0\overline{uw}-\mu\overline{h_x h_z}]$$

is positive and for the second solution it is negative. So the first solution is associated with the upward transfer of energy and the second with the downward transfer of energy.

Similarly, the total momentum flux far from the critical levels is

$$\rho_0\overline{uw}-\mu\overline{h_x h_z}=\begin{cases}-\dfrac{\rho_0\mu_0}{2k}[|A_4|^2-|B_4|^2] & \text{when} \quad z>z_0,\\[2ex] \dfrac{\rho_0\mu_0}{2k}[|A_4|^2e^{2\mu_0\pi}-|B_4|^2e^{-2\mu_0\pi}] & \text{when} \quad z<z_0.\end{cases} \tag{5.5a}$$

The magnitudes of each term in (5.5*a*) at a given distance above and below the critical level $\Omega_d = 0$ are not the same but differ by a factor $\exp\{-2\mu_0\pi\}$, which is exactly the same as the hydrodynamic result of BB. Thus the waves are attenuated by a factor $\exp\{-2\mu_0\pi\}$ and the influence of magnetic field on the absorption of waves at the critical levels is negligible. This process of critical-layer absorption depends only on the gross features of the situation and not on the details of the critical layer.

Now we look at the amplitude of the growing wave motion (i.e. $c_i > 0$). For the upward-travelling wave the amplitude below the critical layer at z_2 (i.e. where $\Omega_d = -\Omega_A$) is $|A_3|\,|z-z_0|^{\frac{1}{2}}\exp\{\mu_m\pi\}$ and above the critical layer at z_1 (i.e. where $\Omega_d = \Omega_A$) is $|A_3|\,|z-z_0|^{\frac{1}{2}}$, which is substantially smaller than $|A_3|\,|z-z_0|^{\frac{1}{2}}\exp\{\mu_m\pi\}$. Hence the amplitude of the growing wave decreases as z increases. Similarly, for the downward-travelling wave the amplitude decreases with decreasing z.

It is of interest to know the magnitude of the velocity and energy at the critical layers when $c_i = 0$. For the solution with $c_i = 0$ both the vertical velocity (w) and the horizontal velocity ($u = \mathrm{Re}\,[(i/k)\,dw/dz]$) are large near the critical level $\Omega_d = \Omega_A$, where w varies as $\log(z-z_1)$ and u varies as $(z-z_1)^{-1}$, whereas in the hydrodynamic case of BB w is small and u is large near the critical level $\Omega_d = 0$. This is consistent with the particle motion becoming more and more horizontal as the wave approaches $z = z_1$; the kinetic and magnetic energies are concentrated entirely in the horizontal motion, but the potential energy is still associated with vertical displacements, so the wave frequency tends to zero. Both the horizontal wave energy per unit volume and the shear associated with the wave vary as

$(z-z_1)^{-2}$. However, if the phase velocity c is complex there will be no singularity at $z = z_1$ and none of these quantities will become infinite at z_1. The same behaviour will be observed at $\Omega_d = -\Omega_A$ but at $\Omega_d = 0$ the magnitudes will be small. Far from the critical levels the solution is given by (3.4). From this it follows that, since the vertical and horizontal velocities vary as $(z-z_0)^{\frac{1}{2}}$ and $(z-z_0)^{-\frac{1}{2}}$ respectively, the horizontal wave energy per unit volume varies as $(z-z_0)^{\frac{1}{2}}$ and the shear stress associated with the wave varies as $(z-z_0)^{-\frac{3}{2}}$.

5.2. *Group velocity near critical levels*

In the previous section we discussed the phenomenon of absorption of waves at the critical levels using the concept of momentum transport. An alternative description of this process, following Bretherton (1966), can be given by using the concept of group velocity. In a slowly varying media an internal Alfvén–gravitational wave with horizontal wavenumber k and vertical wavenumber m satisfies a dispersion relation

$$\sigma = kU \pm \{\Omega_A^2 + k^2N^2/(m^2+k^2)\}^{\frac{1}{2}}. \tag{5.6}$$

We consider a wave packet as a localized disturbance with a reasonably well defined dominant frequency and wavenumber. This wave packet moves with group velocity

$$\mathbf{c}_g = (\partial\sigma/\partial k, \partial\sigma/\partial m).$$

As the wave packet approaches the upper critical level (i.e. $\Omega_d \rightarrow \Omega_A$), from (5.6), we have $m \rightarrow \infty$, i.e. the wavelength $2\pi/m \rightarrow 0$, and hence the wave fronts become more and more horizontal. Then there will be no wave motion and the vertical group velocity

$$\frac{\partial\sigma}{\partial m} = \mp\left\{\Omega_A^2 + \frac{k^2N^2}{m^2+k^2}\right\}^{-\frac{1}{2}} \frac{mk^2N^2}{(m^2+k^2)^2} \tag{5.7}$$

decreases to zero as wave approaches $\Omega_d = \Omega_A$. Near this level $z = z_1$, at which $\Omega_d - \Omega_A$ vanishes, (5.7) can be written as

$$\partial\sigma/\partial m \sim (k/N)(U-c)^{\frac{1}{2}}\, U_z^{\frac{3}{2}}(z-z_1)^{\frac{3}{2}}. \tag{5.8}$$

Thus, when $z-z_1$ is small the height z of the wave packet satisfies

$$dz/dt = b(z-z_1)^{\frac{3}{2}},$$

where b is a constant. On integrating this equation we obtain

$$(z-z_1)^{\frac{1}{2}} = -2/bt. \tag{5.9}$$

In the hydrodynamic case of Bretherton (1969) this equation is of the form

$$z-z_c = -1/at,$$

where the critical level z_c is at $\Omega_d = 0$, whereas in our case the critical level z_1 is at $\Omega_d = \Omega_A$.

From (5.9) it follows that the time taken by a downward-travelling group to pass from a level z_3 to a level z_4 is

$$t_2-t_1 \sim \frac{2N}{k(U-c)^{\frac{1}{2}}\, U_z^{\frac{3}{2}}}\left[\frac{1}{(z_4-z_1)^{\frac{1}{2}}} - \frac{1}{(z_3-z_1)^{\frac{1}{2}}}\right], \tag{5.10}$$

which is arbitrarily large if z_4 is sufficiently near z_1. In other words, the wave packet takes an infinite time to reach the critical level. A wave group will thus never reach the critical level. So the wave packet is neither transmitted nor reflected and simply slows down until either diffusion, turbulence or other non-linearities destroy it. Similar results can be obtained near the critical level $z = z_2$. However, near the critical level $z = z_0$, i.e. $\Omega_d = 0$, m is imaginary; in fact m is imaginary for all $|\Omega_d| < \Omega_A$. This region $|\Omega_d| < \Omega_A$, as can be seen from (5.6), is the forbidden zone for propagation of waves.

The above analysis is confined to $\Omega_d^2 < \Omega_A^2 + N^2$. However, it is of interest to know the corresponding results when $\Omega_d^2 \geqslant \Omega_A^2 + N^2$. In this case, let $z = z_m$ be such that $\Omega_d^2(z_m) = N^2 + \Omega_A^2$. At $z = z_m$, m becomes very small and if z exceeds z_m, i.e. $\Omega_d^2 > N^2 + \Omega_A^2$, m becomes imaginary. Thus the total internal reflexion occurs at $z = z_m$ as the relative frequency $(\Omega_d^2 - \Omega_A^2)^{\frac{1}{2}}$ of an internal Alfvén–gravitational wave cannot exceed N, and the position of the point of reflexion is changed with respect to that for the hydrodynamic case. In this neighbourhood

$$m^2 \sim \pm \frac{2k^3(N^2+\Omega_A^2)^{\frac{1}{2}}}{N^2} U_z(z-z_m),$$

where the sign is positive if z_m is a minimum and negative if it is a maximum. Then $\partial\sigma/\partial m$ is proportional to $\pm |z-z_m|^{\frac{1}{2}}$ and the time taken for the wave group to reach z_m and be reflected is finite.

6. The time-dependent disturbance above a sinusoidal corrugation

In this section, the general results of §§ 3–5 are illustrated by considering a particular problem similar to that of BB. We consider a constant shear flow (see (6.1) below) with a uniform applied magnetic field H_0 parallel to the flow. Since the fluid is perfectly conducting and non-viscous and the applied magnetic field is uniform, the basic velocity is independent of the magnetic field. We take N^2 to be independent of the height z, and the basic velocity $U(z)$ in the x direction is as shown in figure 1:

$$U(z) = \begin{cases} U'(z-h) & (0 < z < 2h), \\ U'h & (z > 2h). \end{cases} \tag{6.1}$$

The fluid is unbounded above and initially at rest everywhere:

$$w = 0 \quad \text{everywhere for} \quad t < 0. \tag{6.2}$$

At time $t = 0$ the disturbance is introduced by raising a sinusoidal corrugation on the lower boundary at $z = 0$, and subsequently maintaining it:

$$w = a\cos kx \quad \text{on} \quad z = 0 \quad \text{for} \quad t > 0, \tag{6.3}$$

where a is the amplitude of the corrugation. The upper boundary condition is

$$w \to 0 \quad \text{as} \quad z \to \infty \quad \text{for} \quad t > 0. \tag{6.4}$$

The perturbation $w(x, z, t)$ satisfies the wave equation (2.2), which was obtained on the basis of a linearization which is valid only for small amplitude a. However,

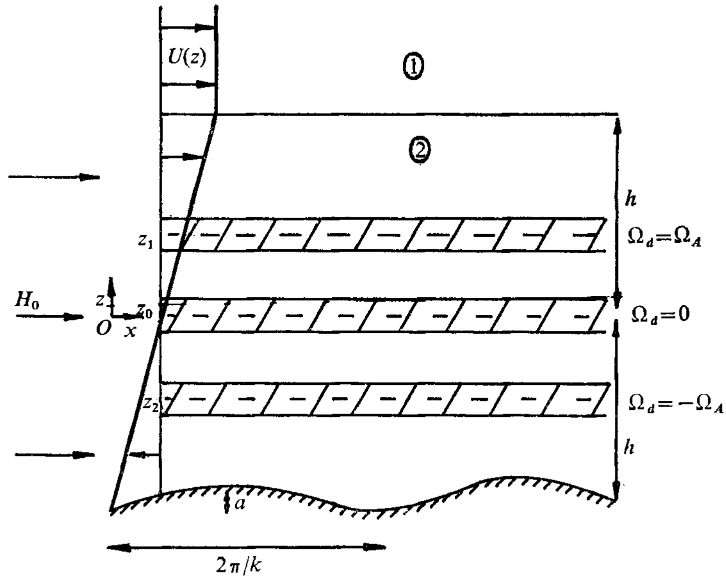

FIGURE 1. The basic state. – – –, critical levels; ////, critical layers.

for any amplitude, it ultimately breaks down. U_{zz} vanishes everywhere except at $z = 2h$, where it becomes infinite and is replaced by a delta function:

$$U_{zz} = -U'\delta(z-2h). \tag{6.5}$$

This is equivalent to matching the pressure and vertical velocity across the perturbed interface between two separate fluids in regions 1 and 2.

We now introduce the dimensionless variables

$$\left.\begin{aligned} \xi = x/h, \quad \zeta = (z-h)/h, \quad \tau = U't, \quad K = kh,\\ \gamma = c/U'h, \quad S = A/U'h, \quad J = N^2/U'^2+K^2S^2,\end{aligned}\right\} \tag{6.6}$$

where K is the dimensionless wavenumber, γ is the dimensionless phase velocity, S is the Alfvén number and J is the modified total Richardson number. Use of the sinusoidal variation in ξ and the Laplace transform in time yields a convenient solution of wave equation (2.2). Let

$$\left.\begin{aligned} w(\xi,\zeta,\tau) &= \mathrm{Re}\,[\tilde{w}(\zeta,\tau)\,e^{iK\xi}],\\ \hat{w}(\zeta,\gamma) &= \frac{1}{(2\pi)^{\frac{1}{2}}}\int_0^\infty \tilde{w}(\zeta,\tau)\,e^{iK\gamma\tau}\,d\tau.\end{aligned}\right\} \tag{6.7}$$

For the convergence of this integral, we assume that the relevant part of the complex plane corresponds to $\gamma_i > 0$. By applying the Laplace transform to the wave equation (2.2) we get two equations corresponding to region 1 and 2:

$$[(1-\gamma)^2-S^2]\,\hat{w}_{\zeta\zeta}+[J-K^2(1-\gamma)^2]\,\hat{w} = 0 \quad \text{in} \quad \zeta>1 \quad \text{for region 1}, \tag{6.8}$$

$$[(\zeta-\gamma)^2-S^2]\,\hat{w}_{\zeta\zeta}+\frac{2S^2}{\zeta-\gamma}\hat{w}_\zeta+\left[J-K^2(\zeta-\gamma)^2-\frac{2S^2}{(\zeta-\gamma)^2}\right]\hat{w} = 0$$

$$\text{in} \quad -1<\zeta<1 \quad \text{for region 2}. \tag{6.9}$$

Each of these equations reduces to the corresponding one in BB in the limit $S \to 0$.

Now we need suitable boundary conditions. The continuity of pressure across the interface between the regions 1 and 2 yields

$$\hat{w}_{1\zeta} - \hat{w}_{2\zeta} + \hat{w}/(1-\gamma) = 0 \quad \text{at} \quad \zeta = 1, \tag{6.10}$$

which is same as the hydrodynamic boundary condition of BB. The continuity of vertical velocity gives

$$\hat{w}_1 = \hat{w}_2 \quad \text{at} \quad \zeta = 1. \tag{6.11}$$

The boundary conditions (6.3) and (6.4) now take the form

$$\hat{w} \to 0 \quad \text{as} \quad \zeta \to \infty, \tag{6.12}$$

$$\hat{w} = -\frac{a}{(2\pi)^{\frac{1}{2}}} \frac{1}{iK\gamma} \quad \text{on} \quad \zeta = -1. \tag{6.13}$$

From (6.13) it is seen that $\gamma = 0$ is a pole which arises because of the specific time dependence assumed in (6.3). If the forcing at $z = 0$ is removed after a finite time, the boundary condition will have no singularity in the complex-γ plane. In the corresponding problem of hydrodynamics, Booker & Bretherton have obtained the solution of the wave equation in terms of modified Bessel functions. However, in the present analysis we cannot obtain the solution of the wave equation in terms of Bessel functions but the solution can be obtained using the Frobenius method.

In region 2, the solution of wave equation (6.9) near $\zeta = \gamma$ is

$$\hat{w} = A_1(\zeta-\gamma)^2 I_1(\zeta-\gamma) + B_1(\zeta-\gamma) J_1(\zeta-\gamma), \tag{6.14}$$

where A_1 and B_1 are constants of integration,

$$I_1(\zeta-\gamma) = 1 + a_2(\zeta-\gamma)^2 + \ldots,$$
$$J_1(\zeta-\gamma) = 1 + b_2(\zeta-\gamma)^2 + \ldots,$$
$$a_2 = (2+J)/6S^2, \quad b_2 = J/2S^2.$$

The solution near the critical level $\zeta = \gamma + S$ is

$$\hat{w} = [A_2 + B_2 \log(\zeta-\gamma-S)] I_2(\zeta-\gamma-S) + B_2 J_2(\zeta-\gamma-S), \tag{6.15}$$

where A_2 and B_2 are constants of integration,

$$I_2(\zeta-\gamma-S) = 1 + c_1(\zeta-\gamma-S) + \ldots,$$
$$J_2(\zeta-\gamma-S) = 1 + d_1(\zeta-\gamma-S) + \ldots,$$
$$c_1 = \tfrac{1}{2}(J - K^2S^2 - 2), \quad d_1 = \tfrac{1}{2}(J - K^2S^2 + 7).$$

A similar solution can be obtained near the critical level $\zeta = \gamma - S$.

In region 2, away from these three singularities, the solution of (6.9), neglecting $K^2(\zeta - S)^2$ compared with J (i.e. assuming that the Doppler-shifted frequency is very much smaller than the algebraic sum of the Brunt–Väisälä frequency N and the Alfvén frequency Ω_A), is

$$\hat{w} = A_3(\zeta-\gamma)^{\frac{1}{2}+i\mu_m} I_3(\zeta-\gamma) + B_3(\zeta-\gamma)^{\frac{1}{2}-i\mu_m} J_3(\zeta-\gamma), \tag{6.16}$$

where

$$I_3(\zeta-\gamma) = 1+\frac{m_2}{(\zeta-\gamma)^2}+\dots,$$

$$J_3(\zeta-\gamma) = 1+\frac{q_2}{(\zeta-\gamma)^2}+\dots,$$

$$m_2 = \frac{(1+r_1)(2+r_1)}{2(3+2r_1)}S^2, \quad q_2 = \frac{(1+r_2)(2+r_2)}{2(3+2r_2)}S^2,$$

$$r_1 = -\tfrac{1}{2}-i\mu_m, \quad r_2 = -\tfrac{1}{2}+i\mu_m.$$

In region 1, the solution of (6.8) consistent with the upper boundary condition (6.12) is

$$\hat{w} = A_4 e^{il(\zeta-1)}, \tag{6.17}$$

where

$$l = \left[\frac{J-K^2(1-\gamma)^2}{(1-\gamma)^2-S^2}\right]^{\frac{1}{2}}, \tag{6.18}$$

with $l_i > 0$ when $\gamma_i > 0$.

The constants A_3, B_3 and A_4 in (6.16) and (6.17) can, by using the boundary conditions (6.10), (6.11) and (6.13), be determined in the form

$$A_3 = \frac{a}{Q(2\pi)^{\frac{1}{2}}}\frac{1}{iK\gamma}\frac{(1-\gamma)^{-i\mu_m}}{(-1-\gamma)^{\frac{1}{2}}} \times\{[1+il(1-\gamma)]J_3(1-\gamma)-(1-\gamma)J_3'(1-\gamma)-(\tfrac{1}{2}-i\mu_m)J_3(1-\gamma)\}, \tag{6.19}$$

$$B_3 = \frac{a}{Q(2\pi)^{\frac{1}{2}}}\frac{1}{iK\gamma}\frac{(1-\gamma)^{i\mu_m}}{(-1-\gamma)^{\frac{1}{2}}} \times\{[1+il(1-\gamma)]I_3(1-\gamma)-(1-\gamma)I_3'(1-\gamma)-(\tfrac{1}{2}+i\mu_m)I_3(1-\gamma)\}, \tag{6.20}$$

$$A_4 = \frac{a}{Q(2\pi)^{\frac{1}{2}}}\frac{1}{iK\gamma}\left(\frac{1-\gamma}{-1-\gamma}\right)^{\frac{1}{2}} \times\{[I_3(1-\gamma)J_3'(1-\gamma)-J_3(1-\gamma)I_3'(1-\gamma)](1-\gamma)-2i\mu_m I_3(1-\gamma)J_3(1-\gamma)\}, \tag{6.21}$$

where

$$\begin{aligned} Q = {} & (-1-\gamma)^{i\mu_m}(1-\gamma)^{-i\mu_m}\{(1-\gamma)I_3(-1-\gamma)J_3'(1-\gamma) \\ & +(\tfrac{1}{2}-i\mu_m)I_3(-1-\gamma)J_3(1-\gamma)+[1+il(1-\gamma)]I_3(1-\gamma)J_3(-1-\gamma)\} \\ & -(-1-\gamma)^{-i\mu_m}(1-\gamma)^{i\mu_m}\{(1-\gamma)J_3(-1-\gamma)I_3'(1-\gamma) \\ & +(\tfrac{1}{2}+i\mu_m)J_3(-1-\gamma)I_3(1-\gamma)+[1+il(1-\gamma)]J_3(1-\gamma)I_3(-1-\gamma)\}, \end{aligned} \tag{6.22}$$

and the primes on I_3 and J_3 denote derivatives with respect to ζ.

The complete solution to the problem is

$$w(\xi,\zeta,\tau) = \mathrm{Re}\left[e^{iK\xi}\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\hat{w}(\gamma,\zeta)\,e^{i-K\gamma\tau}K\,d\gamma\right], \tag{6.23}$$

where the path of integration lies along the real axis but is deformed to pass above the singularities of the integrand (i.e. $\gamma_i > 0$).

7. Wave propagation after a long time

Equation (6.23) is in general mathematically complicated and will be simplified if τ is large. In this case we use the 'method of steepest descents' (Jeffreys & Jeffreys 1946, § 17.04) to integrate (6.23). The value of the integral can be obtained by considering the neighbourhoods in the complex-γ plane of points where either the integrand is singular or the derivative with respect to γ of the coefficient of τ in the exponent in (6.23) vanishes (saddle point). If there are no saddle points the largest contribution comes from the pole at $\gamma = 0$.

In the remaining portion of this section, using the above method, we show that if the total Richardson number $J \geqslant 1$ the waves are very much reduced in magnitude. Away from the critical levels, the motion becomes that of a standing wave pattern, there being several small decaying wave motions superposed on this steady wave motion. There are regions above and below each of the critical levels $\zeta = \zeta_1$ and $\zeta = \zeta_2$ called critical layers which decrease in thickness. In these two critical layers, the horizontal momentum associated with the wave is nearly all transferred to the mean flow. In the region around the critical level $\zeta = \zeta_0$ there is no absorption of waves. If ζ/τ is kept constant as $\tau \to \infty$, as in the case of BB, the largest contribution comes from the saddle point. This corresponds to an upward-travelling dispersing group of waves and its dominant frequency is such that the corresponding vertical component of group velocity is ζ/τ.

The above results can be proved by considering the singularities of the integrand in the integral in (6.23). These are (*a*) $\gamma = 0$, a pole, arising from the applied boundary condition (6.3); (*b*) $\gamma = \pm 1$, branch points; (*c*) $\gamma = 1 \pm S$, essential singularities, $l \to \infty$; (*d*) $\gamma = 1 \pm J/K$, branch points, $l = 0$; (*e*) $\gamma = \zeta \pm S$, branch points. In addition there may be poles where $Q(\gamma) = 0$. The singularity (*a*) is similar to that of the hydrodynamic case of BB whereas other singularities differ from the hydrodynamic case because of the presence of the magnetic field. The singularities (*e*) are due to the occurrence of the logarithmic terms in the solutions near the critical levels $\Omega_d = \pm \Omega_A$. From a theorem (an extension of Miles' (1961) theorem to MHD) proved by Narayan (1972) it follows that all these singularities will lie in $\gamma_i \leqslant 0$ provided $J > \frac{1}{4}$, and we consider only these exponentially vanishing wave motions in our analysis. A zero of Q in $\gamma_i > 0$ corresponds to an exponentially growing mode; the zeros of Q can be found explicitly and are not relevant to the present analysis.

We assume that neither of the singularities (*e*) coincide with any other singularities, and the path of integration Σ is deformed according to figure 2 so that all the singularities lie in the region $\gamma_i < 0$. As $\tau \to \infty$ the integrand becomes exponentially small, except in the regions near the real axis $\gamma_i = 0$.

The largest contribution to the integral (6.23) comes from the pole at $\gamma = 0$ and is equal to $2\pi i$ times the residue at this pole. Thus at large times, i.e. as $\tau \to \infty$, equation (6.23) takes the form

$$w(\xi, \zeta, \tau) \to \mathrm{Re}\,\{(2\pi)^{\frac{1}{2}}\, iK \lim_{\gamma\to 0}\,[\gamma \hat{w}(\gamma)\, e^{iK\xi}]\}. \tag{7.1}$$

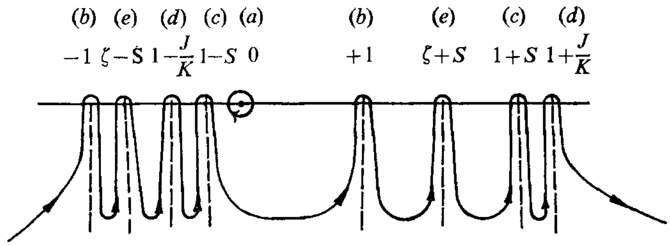

FIGURE 2. The deformed contour of integration.

In region 1 this reduces to

$$w \sim \mathrm{Re}\left\{(2\pi)^{\frac{1}{2}} iKA_4'(0)\exp\left[i\left(\frac{J-K^2}{1-S^2}\right)^{\frac{1}{2}}(1+\zeta)+iK\xi\right]\right\}, \tag{7.2}$$

where $$A_4'(0) = \lim_{\gamma\to 0}[\gamma A_4(\gamma)].$$

This is a stationary upward-propagating wave (see § 4). In region 2, away from the critical levels, we have

$$w \sim \mathrm{Re}\,(e^{iK\xi}\{(2\pi)^{\frac{1}{2}} iK[A_3'(0)\,\zeta^{\frac{1}{2}+i\mu_m}I_3(\zeta)+B_3'(0)\,\zeta^{\frac{1}{2}-i\mu_m}J_3(\zeta)]\}), \tag{7.3}$$

where $$A_3'(0) = \lim_{\gamma\to 0}[\gamma A_3(\gamma)],\quad B_3'(0) = \lim_{\gamma\to 0}[\gamma B_3(\gamma)].$$

The first term in (7.3) describes an upward-travelling wave and the second a downward-travelling wave. For an upward-travelling wave the wave amplitude is reduced by a factor $\exp(-\mu_m\pi)$ above the upper critical level $\zeta = \zeta_1$ from its value below the lower critical level $\zeta = \zeta_2$. If μ_m is very large the waves are completely attenuated.

If ζ/τ is kept constant and positive as $\tau\to\infty$ the saddle point for the exponent in (6.23) is given by

$$\frac{1}{K}\frac{d}{d\gamma}\left[\frac{J-K^2(1-\gamma)^2}{(1-\gamma)^2-S^2}\right]^{\frac{1}{2}} = \tau. \tag{7.4}$$

However, ζ/τ is the group velocity of a wave of frequency $K\gamma$ (Jeffereys & Jeffereys 1946, § 17.08), hence (7.4) is the condition that the vertical component of group velocity of a wave of frequency $K\gamma$ in the uniform region 1 should equal ζ/τ. The integration by the saddle-point method contributes to the integral, the resulting wave, with frequency equal to that at the saddle point, decreasing in amplitude like $\tau^{-\frac{1}{2}}$.

Using the Riemann–Lebesgue lemma it can be shown that the contributions from the other singularities vanish as $\tau\to\infty$. However, we examine in detail the contribution from one of the branch points (*e*), i.e. $\gamma = \zeta - S$. We put $\gamma = \zeta - S + \delta$ to find the contribution from the neighbourhood of $\gamma = \zeta - S$. The integrand is expanded as a power series in δ and we consider only the leading term, which is of the form

$$w(\xi,\zeta,\tau) = \mathrm{Re}\left[\frac{1}{(2\pi)^{\frac{1}{2}}}K\exp\left[iK(\xi-\zeta\tau+S\tau)\right]\int_{\Sigma_e}\{A_2(\zeta)+B_2(\zeta)+B_2\log(-\delta)\}\,d\delta\right]. \tag{7.5}$$

The path of integration Σ_e is that portion of the deformed contour near the point $\gamma = \zeta - S$, and $A_2(\zeta)$ and $B_2(\zeta)$ are the values of A_2 and B_2 when $\gamma = \zeta - S$. On integrating (7.5) by the method of contour integration, we obtain

$$w(\xi, \zeta, \tau) = \mathrm{Re}\{(1/\tau)(2\pi)^{\frac{1}{2}} \exp[iK(\xi - \zeta\tau + S\tau)] \times [A_2 + B_2 - \tfrac{1}{2}iB_2\pi - B_2\psi(-1) + \log(K\tau)]\}, \quad (7.6)$$

where ψ is the digamma function, defined by $\psi(z) = \Gamma'(z)/\Gamma(z)$, $\Gamma(z)$ being the gamma function (Abramowitz & Stegun 1965, §6.3). For the hydrodynamic case Booker & Bretherton have expressed the solution in terms of the modified Bessel function. Note that in the corresponding problem in hydromagnetics discussed here the solution involves a digamma function.

Equation (7.6) describes locally plane sinusoidal waves with phase

$$K(\xi - \zeta\tau + S\tau).$$

This phase function is different from that for the hydrodynamic case of BB. As in hydrodynamics, the lines of constant phase are advected with the basic velocity and the phase front becomes nearly horizontal. Differentiation of the phase function with respect to ξ and ζ respectively shows that the horizontal wavenumber remains constant and the vertical wavenumber increases with time. From (7.6) it follows that the vertical velocity decays as τ^{-1} and the horizontal velocity increases logarithmically as $\log(K\tau)$. The vertical energy density decays as τ^{-2} whereas the horizontal energy density increases with time. The Reynolds stress is different above and below each critical level and the oscillations are absorbed into the mean flow. This absorption is associated with the continuum distribution of the disturbance over a band of frequencies, each frequency having a distinct critical level. In the hydrodynamic problem of BB, both the vertical and horizontal energy densities decay with time whereas in the hydromagnetic problem discussed here the vertical energy density decreases with time but the horizontal energy density increases logarithmically. There are other waves contributed by the singularities (*b*), (*c*) and (*d*) and all these waves decay to zero as $\tau \to \infty$.

In the above discussion we have assumed that the singularities (*a*)–(*e*) are distinct. Even when the singularity (*e*) coincides with (*a*), (*b*), (*c*) or (*d*) equation (6.23) can be integrated and the corresponding wave decays to zero as $\tau \to \infty$. However, the region in which $\zeta \to S$ as $\tau \to \infty$ (critical layer) is of more interest. When the singularity $\gamma = \zeta - S$ coincides with $\gamma = 0$ the contributions from these singularities can no longer be separated and their neighbourhoods must be treated together. Now the integrand is expanded as a power series in $\gamma - \zeta + S$, assuming ζ is also small, and if we put $\gamma - \zeta + S = \lambda/K\tau$ the leading term in the equation is

$$w \sim \mathrm{Re}\left[\frac{K\exp[iK(\xi - \zeta\tau + S\tau)]}{(2\pi)^{\frac{1}{2}}} \int_{\Sigma_{ae}} \frac{A_2'(0) + B_2'(0) + B_2'(0)\log(-\lambda/K\tau)}{\lambda + K\tau(\zeta - S)} e^{-i\lambda}\, d\lambda\right], \quad (7.7)$$

where the contour Σ_{ae} is from $\lambda = -i\infty$ to $\lambda = -i\infty$ round both $\lambda = 0$ and $\lambda = -K\tau(\zeta - S)$. Equation (7.7) gives the structure of the velocity distribution.

It is of interest to compare the velocity given by (7.7) with equation (5.9) of Booker & Bretherton (1967). In their analysis the integral is evaluated around $\Omega_d = 0$ and the integrand is a power series in λ. They found that a typical vertical velocity has a magnitude of order $\tau^{-\frac{1}{2}}$ and a typical horizontal velocity is of order $\tau^{\frac{1}{2}}$.† In the present analysis, however, the integral is evaluated in the neighbourhood of the magnetic critical layer $\Omega_d = \Omega_A$ and the integrand is a logarithmic function of λ. A typical vertical velocity has a magnitude of order $\log(K\tau)$ and the horizontal velocity is of order $\tau\log(K\tau)$. Thus both vertical and horizontal velocity components increase with time τ, and ultimately nonlinear terms which have been ignored earlier will become important and the theory will be invalid. However, this invalidation may be delayed by taking a, in (6.3), small enough. We note that this logarithmic singularity in (7.7) is mainly due to the effect of Lorentz force at the critical level $\Omega_d = \Omega_A$. Similar results can be obtained across the critical level $\Omega_d = -\Omega_A$.

Finally, in region 1 we get some interesting results when $K \ll 1$ and $|\zeta - \gamma| > S$, and the motion is everywhere very nearly horizontal. Then in the final steady state $\gamma = 0$, $l \sim J = (\mu_m^2 + \frac{1}{4})^{\frac{1}{2}}$ and $I_3(1-\gamma) = 1$, $J_3(1-\gamma) = 1$. Under these conditions we have

$$\lim_{\gamma\to 0}\left(\frac{\gamma B_3(\gamma)}{\gamma A_3(\gamma)}\right) = \frac{B_3'}{A_3'} = \frac{\frac{1}{2} + i(\mu_m^2 + \frac{1}{4})^{\frac{1}{2}} - i\mu_m}{\frac{1}{2} + i(\mu_m^2 + \frac{1}{4})^{\frac{1}{2}} + i\mu_m}. \tag{7.8}$$

Above the critical layers the ratio of the energy fluxes, or the total stresses, associated with the downward- and upward-travelling waves is

$$\frac{|B_3'|^2}{|A_3'|^2} = \frac{(\mu_m^2 + \frac{1}{4})^{\frac{1}{2}} - \mu_m}{(\mu_m^2 + \frac{1}{4})^{\frac{1}{2}} + \mu_m} \sim \frac{1}{16\mu_m^2}. \tag{7.9}$$

Thus, if μ_m is large, very little energy is reflected by the discontinuity in U_z at the interface between the regions 1 and 2. Below the critical layer $\Omega_d = -\Omega_A$ the difference in the energy fluxes is even larger, being

$$\frac{|B_3'|^2}{|A_3'|^2} e^{-4\mu_m\pi} \sim \frac{1}{16\mu_m^2} e^{-4\mu_m\pi}. \tag{7.10}$$

Hence, if $\mu_m > 1$ the effect of the region above the critical layer ($\Omega_d > \Omega_A$) on the region below ($\Omega_d < -\Omega_A$) is almost negligible. Thus the critical layers act as an absorbing barrier of great effectiveness. On comparing the corresponding hydrodynamic results of BB with the present results (7.9) and (7.10), we find that the effect of Lorentz force at the critical layers is to increase the absorption (since $\mu_m > \mu_0$) and to lessen the reflexion by the discontinuity in U_z at the interface between regions 1 and 2.

8. Transient disturbances in perfectly conducting shear flows

Although a similar analysis to that of a disturbance above a sinusoidal corrugation discussed in §§6 and 7 may be used to describe the disturbance due to a transient stimulus in a shear layer, the resulting integral is mathematically

† In BB there are misprints in the order of τ, i.e. $\tau^{\frac{1}{2}}$ and $\tau^{-\frac{1}{2}}$ are interchanged. The above forms are the correct ones.

complicated and cannot be evaluated in terms of elementary functions. However, following BB, we can make some general statements about the velocity distribution.

The asymptotic solution for the velocity distribution discussed in §§ 6 and 7 has a pole at $\gamma = 0$ because of the assumed lower boundary condition $w = a\cos kx$. However, in the present analysis there will be no pole at $\gamma = 0$. Hence the dominant contributions come from the singularities of the type (*b*)–(*e*) and decay with time. The singularities (*b*), (*c*) and (*d*) arise because of the velocity profile assumed in § 6 and would not be present in an unbounded uniform shear flow.

In the present case the velocitities associated with the singularity (*e*), as in § 7, can be shown to be of the form

$$w = \mathrm{Re}\left[\left(\frac{F}{t}+\frac{G\log(kt)}{t}\right)\exp\{ik[x-U(z)\,t+At]\}\right]. \tag{8.1}$$

Each term in (8.1) describes locally plane waves of very small vertical wavelength $-1/2\pi ktU_z$, which decays as t^{-1}. The horizontal disturbance is given by

$$u = \frac{i}{k}\frac{\partial w}{\partial z} = \mathrm{Re}\{[F+G\log kt]\,U_z\exp[ik(x-Ut+At)]\}, \tag{8.2}$$

which increases logarithmically with time. We find that the decay of the vertical velocity field of the form (8.1) is a manifestation of critical-layer absorption for a continuum spectrum of frequencies. Each frequency is associated with one of the critical levels z_1, z_0 and z_2 and at each height z there are corresponding frequencies σ_1, σ_0 and σ_2 for which this height is critical. A qualitative explanation of this absorption effect is provided by the concept of group velocity discussed in § 5.2.

We now check the validity of the linearization assumed in this paper, using (8.1) and (8.2). The nonlinear terms $u\,\partial w/\partial x + w\,\partial w/\partial z$ and $h_x\,\partial h_z/\partial x + h_z\,\partial h_z/\partial z$ are of order F^2t^{-1}, whereas the linear terms $\partial w/\partial t + U\,\partial w/\partial x$ and $\partial h_z/\partial t + H_0\,\partial h_z/\partial x$ are of order Ft^{-2}. Thus, in the absence of dissipation, the nonlinear terms should be taken into account after a time of order F^{-1} however small F may be. By this time the velocity gradients

$$u_z = ktFU_z^2\exp[ik(x-Ut+At)] \tag{8.3}$$

have become comparable with U_z, so the flow may become turbulent. However the vertical scale of the perturbation is then of order $k^{-1}F$, therefore any turbulence may be expected to be of low intensity since F can be made small. Also, by this time the waves are almost completely absorbed by the mean flow, so the main results of this paper are justified. Even for the second terms of (8.1) and (8.2) the validity of linearization can be justified.

Finally, we can obtain an expression for the total change of momentum of the mean flow as a result of the absorption of the disturbance. We consider the total velocity and the magnetic field in the form

$$U(z)+U_1(x,z,t)+u(x,z,t), \quad W_1(x,z,t)+w(x,z,t),$$
$$H_0+H_x(x,z,t)+h_x(x,z,t), \quad H_z(x,z,t)+h_z(x,z,t).$$

Here the perturbations u, w, h_x, h_z and U_1, W_1, H_x and H_z are assumed to vanish rapidly at all finite times so that their integrals converge. U_1, W_1, H_x and H_z

may roughly be described as second-order disturbances associated with first-order small amplitude perturbations u, w, h_x and h_z of zero mean. However, their separation from u, w, h_x and h_z need be made precise only to the extent of specifying

$$\int_{-\infty}^{+\infty} u\,dx = \int_{-\infty}^{+\infty} w\,dx = \int_{-\infty}^{+\infty} h_x\,dx = \int_{-\infty}^{+\infty} h_z\,dx = 0. \tag{8.4}$$

Then we obtain from the continuity equation for the fluid and the continuity of magnetic field that, if W_1 and H_z vanish at some level,

$$\int_{-\infty}^{+\infty} W_1\,dx = \int_{-\infty}^{+\infty} H_z\,dx = 0 \tag{8.5}$$

at all levels. The integral $\int_{-\infty}^{+\infty} U_1\,dx$ does not necessarily vanish and can be identified as the mean-flow total momentum at that level (value of z) associated with the disturbance; we calculate its total change during the absorption process.

The equation for the horizontal velocity is

$$\frac{\partial}{\partial t}(U+U_1+u)+\frac{\partial}{\partial x}(U+U_1+u)^2+\frac{\partial}{\partial z}(U+U_1+u)(w_1+w)$$
$$+\frac{1}{\rho_0}\frac{\partial P}{\partial x}-\frac{\mu}{\rho_0}\frac{\partial}{\partial x}(H_0+H_x+h_x)^2-\frac{\mu}{\rho_0}\frac{\partial}{\partial z}(H_0+H_x+h_x)(H_z+h_z) = 0. \tag{8.6}$$

Since P, u, w and U_1 are assumed to vanish rapidly as $|x| \to \infty$, on integration equation (8.6) gives

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} U_1\,dx = -\frac{\partial}{\partial z}\int_{-\infty}^{+\infty}\left[(U_1+u)(w_1+w)-\frac{\mu}{\rho_0}(H_x+h_x)(H_z+h_z)\right]dx. \tag{8.7}$$

If the disturbances to the basic flow $U(z)$ and the magnetic field H_0 are of small amplitude we can assume that $U_1 \ll u$, $W_1 \ll w$, $H_x \ll h_x$ and $H_z \ll h_z$. Then we have

$$\left[\int_{-\infty}^{+\infty} U_1\,dx\right]_{t=0}^{\infty} = -\int_0^{\infty} dt\int_{-\infty}^{+\infty} dx\,\frac{\partial}{\partial z}\left(uw-\frac{\mu}{\rho_0}h_x h_z\right). \tag{8.8}$$

By expressing the disturbance as a double integral over a continuum of wavenumbers and horizontal phase velocities in the form

$$f(x,z,t) = \mathrm{Re}\left[\frac{1}{\pi}\int_0^{\infty} dk\int_{-\infty}^{+\infty} dc\,\hat{f}(k,z,c)\,e^{ik(x-ct)}\right] \tag{8.9}$$

and using Parseval's theorem, we obtain

$$\int_0^{\infty} dt\int_{-\infty}^{+\infty} dx\,\frac{\partial}{\partial z}\left(uw-\frac{\mu}{\rho_0}h_x h_z\right) = \int_{-\infty}^{+\infty} k\,dc\int_0^{\infty} dk\,\frac{\partial}{\partial z}\left(\overline{uw-\frac{\mu}{\rho_0}h_x h_z}\right), \tag{8.10}$$

where $$\overline{uw-(\mu/\rho_0)\,h_x h_z} = \tfrac{1}{2}\,\mathrm{Re}\,[uw^*-(\mu/\rho_0)\,h_x h_z^*]$$

is the total stress (i.e. the algebraic sum of the Reynolds stress and the magnetic stress) for each Fourier component as computed in § 5. Thus the total momentum transferred upwards past each level is the integral over all wavenumbers and frequencies of the total stress associated with each Fourier component separately.

We have proved earlier, in §5, that the total stress is conserved everywhere except at the critical levels. Hence the only contribution to the right-hand side of (8.10) at any given height z comes from those components for which

$$c = U(z) \pm A. \tag{8.11}$$

So the integration with respect to c is equivalent to integration with respect to z and hence, from (8.8) and (8.10), we have

$$\left[\int_{-\infty}^{+\infty} U_1\, dx\right]_{t=0}^{\infty} = -U_z \int_0^{+\infty} k\, dk \left[\overline{uw - \frac{\mu}{\rho_0} h_x h_z}\right]_-^+, \tag{8.12}$$

where $[\overline{uw-(\mu/\rho_0)\,h_x h_z}]_-^+$ is the sum of the discontinuities, across the critical levels, in the total stress associated with those Fourier components with phase velocities defined by (8.11) and wavenumber k. Equation (8.12) states that the total transfer of momentum to the mean flow associated with the passage and partial absorption of a transient disturbance is finite, is distributed over a range of heights and can be calculated from the discontinuities across the critical layers (if any) of each Fourier component separately. If the disturbance is initiated below $z = z_2$ and travels upwards, and if the total Richardson number J is moderately large, the upward-travelling wave is almost completely absorbed in passing through the critical levels $z = z_2$, z_0 and z_1. The sum of the discontinuities is then simply minus the value of the total stress below $z = z_2$, which is in turn, by (4.5), directly connected to the net upward flux of total wave energy at $z = z_2$.

9. Conclusions

The presence of a magnetic field introduces three critical levels of which one ($\Omega_d = 0$) is the hydrodynamic critical level and the other two ($\Omega_d = \pm\,\Omega_A$) are hydromagnetic critical levels. The wave equation (1.1) corresponding to the problem of BB is a simple harmonic type with a singularity at the hydrodynamic critical level, whereas the hydromagnetic wave equation (2.4) of the present problem is in the form of telegraphic equation with three singularities at hydrodynamic and hydromagnetic critical levels. We find that the attenuation of waves takes place only at the magnetic critical levels. That is, the internal Alfvén–gravity waves propagating with a vertical component of group velocity in a perfectly conducting shear flow in the presence of a magnetic field are almost completely absorbed in passing through the critical levels $\Omega_d = \pm\,\Omega_A$ provided that the Richardson number $J_H \geqslant 1$. For $\frac{1}{4} < J_H < 1$ less absorption takes place. In other words the vertical momentum flux, which is an appropriate measure of the magnitude of the waves, is reduced by a factor $\exp\{-2\pi(J_H-\frac{1}{4})^{\frac{1}{2}}\}$ in passing through the critical levels. Comparing this result with the hydrodynamic result of BB, we conclude that the effect of the magnetic field on the attenuation of waves is negligible. This absorption mechanism is independent of viscosity, finite electrical conductivity or other dissipative processes. When this absorption mechanism is studied through the group velocity approach we find that the wave group travelling with the appropriate local group velocity would take an infinite time to reach the critical levels and thus will never reach the critical levels $\Omega_d = \pm\, \Omega_A$. Hence it is neither transmitted nor reflected but is completely absorbed. Near these critical levels the vertical wavelength becomes very small and the motion is entirely in the horizontal direction; there will be no waves in the 'forbidden zone' $|\Omega_d| < \Omega_A$.

In § 7 it was found that the presence of a magnetic field decreases the reflexion of waves at the discontinuity in U_z. Also, we found that away from the critical levels the motion becomes that of a standing wave pattern. The analysis of §§ 7 and 8 shows that the nonlinear terms become important after a finite time. However, by this time the waves are almost completely attenuated. The main conclusion of § 8 is that the total transfer of momentum to the mean flow associated with the passage and partial absorption of a transient disturbance is finite, distributed over a range of heights and can be calculated from the discontinuity across the critical layers (if any) of each Fourier component separately.

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